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Improving the chiral energy–momentum tensor

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Abstract. The energy–momentum tensor for a chiral-invariant theory of massless pseudoscalar mesons is 'improved' by the methods of Callan, Coleman and Jackiw. The improved tensor has a simple relation to the dilation current and nonvanishing triple-derivative terms in the equal-time commutators of its components.

1. Introduction

The notion of an 'improved' energy–momentum tensor was introduced by Callan *et al* (1970) (see also Coleman and Jackiw 1971) to satisfy the requirement that its trace should have finite matrix elements in a renormalizable theory. The improved tensor $\Theta^{\mu\nu}$ is related to the symmetrical Belinfante (1940) tensor $T^{\mu\nu}$ by the addition of an extra term $\Delta^{\mu\nu}$,

$$\Theta^{\mu\nu} = T^{\mu\nu}+\Delta^{\mu\nu}. \tag{1.1}$$

This term is chosen so that it leaves unaltered the Poincaré generators and the dynamics; it is a symmetrical conserved second-rank Lorentz tensor, and is a total divergence. For spinless fields (when of course the Belinfante tensor coincides with the canonical tensor), they show

$$\Delta^{\mu\nu} = -\tfrac{1}{6}(\partial^{\mu}\partial^{\nu}-\eta^{\mu\nu}\Box^{2})\sigma \tag{1.2}$$

where σ is a scalar function of the fields. In $\lambda\phi^4$ theory σ turns out to be

$$\sigma = \phi^{2}. \tag{1.3}$$

The choice of σ was dictated so that the trace

$$\Theta = \eta_{\mu\nu}\Theta^{\mu\nu} \tag{1.4}$$

of the improved tensor is 'soft'. Unlike the trace of the Belinfante tensor which contains field gradients, they have, in the case of $\lambda\phi^4$ theory with mass m,

$$\Theta = m^{2}\phi^{2}. \tag{1.5}$$

This can also be written as

$$\Theta = m\frac{\partial}{\partial m}\Theta^{00}; \tag{1.6}$$

only that term in the energy density Θ^{00} which involves the constant m which has a dimension contributes to the trace.

This property is intimately connected with another feature of the improved tensor discussed by Callan *et al*, namely its connection with the dilation current. Just as Belinfante's tensor enabled the Lorentz generators to be expressed simply (which is not the case with the canonical tensor for a theory with spin), so likewise can the dilation current $D^\mu(x)$ and hence the generator of dilations

$$D(t) = \int D^0(\boldsymbol{x}, t)\, \mathrm{d}^3x \tag{1.7}$$

be given a simple expression. This is

$$D^\mu = \Theta^{\mu\nu}x_\nu. \tag{1.8}$$

Since $\Theta^{\mu\nu}$ is conserved, it is obvious that we have

$$\partial_\mu D^\mu = \Theta. \tag{1.9}$$

We remark that the equal-time commutators for the components of $\Theta^{\mu\nu}$ imply

$$\mathrm{i}[D(t), \Theta^{00}(\boldsymbol{x}, t)] = (x\,.\,\partial+4)\Theta^{00}-\Theta. \tag{1.10}$$

Taken in conjunction with (1.6) this equation simply ensures that the scale dimension of the energy density is 4.

The equal-time commutators of $\Theta^{\mu\nu}$ are also superior to those of $T^{\mu\nu}$. This is because, as shown by Callan *et al*, they correctly include the triple derivative Schwinger terms whose absence was shown by Boulware and Deser (1967) to be incompatible with positivity.

Now all these arguments have been made heretofore in the context of renormalizable theories, notably $\lambda\phi^4$ theory. We would like to see how far they are applicable, at least formally, to the nonrenormalizable case of the chiral-invariant interactions of massless pseudoscalar mesons. As emphasized by Boulware and Deser the most likely origin of the triple derivative terms is in a careful definition of the product of field operators at the same point. Presumably the additional term $\Delta^{\mu\nu}$ in the improved tensor is mimicking the effect of the appropriate limiting procedure. In the chiral theory the problem is further compounded. Not only does one have to cope with products of fields at the same point, but there are nonpolynomial functions of fields. And as yet a further complication there are field gradients which do not even commute with the fields which they multiply. In this paper, we are not concerned with the last-mentioned difficulty. In a recent paper Parish (1973) has shown that the canonical tensor $T^{\mu\nu}$ may be written as

$$T^{\mu\nu} = \tfrac{1}{2}f_\pi^{-2}\delta_{ab}j^{a\kappa}j^{b\lambda}(\delta^\mu_\kappa\delta^\nu_\lambda+\delta^\mu_\lambda\delta^\nu_\kappa-\eta^{\mu\nu}\eta_{\kappa\lambda}). \tag{1.11}$$

This expression in which the Gell-Mann currents (a and b range over both the axial and vector SU(n) labels) appear symmetrized is uniquely determined by the requirement that it is indeed chiral invariant.

What we propose to do is to try to 'improve' this canonical tensor (this is a theory without spin so $T^{\mu\nu}$ is already the symmetrical Belinfante tensor) by adding a term $\Delta^{\mu\nu}$ to it. We will *not* be able to determine $\Delta^{\mu\nu}$ so that the improved tensor has a 'soft' trace: this after all is a very nonrenormalizable theory! But we shall be able to use the connection with dilations, and will find furthermore that there appear triple derivative terms in the appropriate equal-time commutators.

It should be emphasized at the outset that the discussion in this paper is at best heuristic; the theory is nonrenormalizable and none of the subtleties associated with anomalous dimension, etc, have been considered, let alone dealt with.

2. The improved tensor

Our starting point will be the requirement that the dilation current is given by (1.8). This means that $\Delta^{\mu\nu}$ must be chosen to give, for $x_0 = y_0$,

$$\mathrm{i}\left(\int x_\mu \Theta^{\mu 0}(x)\,\mathrm{d}^3x,\ \phi^i(y)\right) = (y\,.\,\partial+1)\phi^i(y). \tag{2.1}$$

From the relationship between the currents and the canonical fields ϕ^i and momenta π_i it is easy to show that for $x_0 = y_0$

$$\mathrm{i}\left(\int x_\mu T^{\mu 0}(x)\,\mathrm{d}^3x,\ \phi^i(y)\right) = y\,.\,\partial\phi^i(y), \tag{2.2}$$

so we require of $\Delta^{\mu 0}$ that, again when $x_0 = y_0$,

$$\mathrm{i}\left(\int x_\mu \Delta^{\mu 0}(x)\,\mathrm{d}^3x,\ \phi^i(y)\right) = \phi^i(y). \tag{2.3}$$

If we also recognize the need for $\Delta^{\mu\nu}$ to be of the form (1.2), this becomes a condition on σ, namely

$$\sigma_{,i}g^{ij} = 2\phi^j. \tag{2.4}$$

The notation $\sigma_{,i}$ means $\partial\sigma/\partial\phi^i$. The matrix g^{ij} is the inverse of g_{ij}, which in turn is the metric on the manifold parametrized by the meson fields ϕ^i. We thus have a set of partial differential equations

$$\sigma_{,i} = 2g_{ij}\phi^j \tag{2.5}$$

for σ: the integrability conditions

$$(g_{ij,k}-g_{ik,j})\phi^i = 0$$

may be satisfied. We give explicit solutions for σ below.

3. The function σ

In this section we show that a function σ of the fields ϕ which satisfies the requirement (2.5) may always be found. Consider first the case of SU(2) ⊗ SU(2). It is well known (Weinberg 1968) that with the definitions

$$\phi^2 = \phi^i\delta_{ij}\phi^j, \tag{3.1}$$

$$h = (f^2+\phi^2)/(f-2\phi^2 f'), \tag{3.2}$$

$$P_{ij} = \delta_{ij}-Q_{ij}, \tag{3.3}$$

$$Q_{ij} = \phi^i\phi^j/\phi^2; \tag{3.4}$$

the metric g_{ij} is given by

$$g_{ij} = f_\pi^{-2}[(f^2+\phi^2)^{-1}P_{ij}+h^{-2}Q_{ij}]. \tag{3.5}$$

There is an arbitrary function $f(\phi^2)$, normalized so that $f(0) = f_\pi$, specification of which corresponds to a particular 'gauge' for the pion fields, ie, to a particular parametrization of the manifold on which g_{ij} is a metric.

Taking $\sigma = \sigma(\phi^2)$, the requirement (2.5) becomes

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\phi^2} = \frac{f_\pi^2}{h^2}, \tag{3.6}$$

which may be integrated for any choice of f. In particular for 'normal coordinates', when

$$f = \sqrt{\phi^2}\cot\sqrt{\phi^2/f_\pi^2}, \tag{3.7}$$

for which choice

$$h = f_\pi, \tag{3.8}$$

one obtains the simple result

$$\sigma = \phi^2. \tag{3.9}$$

Turning now to the case of $\mathrm{SU}(n) \otimes \mathrm{SU}(n)$, but continuing to use normal coordinates, we define an antisymmetric $n \times n$ matrix $\boldsymbol{x}$ linear in the fields ϕ^k through

$$x_{ij} = f_{ijk}\phi^k. \tag{3.10}$$

The quantities f_{ijk} are the conventional (Gell-Mann) totally antisymmetric structure constants of $\mathrm{SU}(n)$. For normal coordinates we have (Callan *et al* 1969, Charap 1970)

$$\boldsymbol{g} = \frac{\sin^2\boldsymbol{x}}{\boldsymbol{x}^2}. \tag{3.11}$$

From the obvious relation

$$x_{ij}\phi^j = 0, \tag{3.12}$$

it follows that

$$g_{ij}\phi^j = \delta_{ij}\phi^j. \tag{3.13}$$

Hence the differential equations (2.5) for σ become

$$\sigma_{,i} = 2\delta_{ij}\phi^j. \tag{3.14}$$

with solution again

$$\sigma = \phi^i\delta_{ij}\phi^j \equiv \phi^2. \tag{3.15}$$

Note that we always have

$$\tfrac{1}{2}\phi^k\sigma_{,k} = \phi^k g_{kl}\phi^l, \tag{3.16}$$

and in normal coordinates

$$\tfrac{1}{2}\phi^k\sigma_{,k} = \phi^2. \tag{3.17}$$

4. The equal-time commutators

Having used the connection with dilations to determine $\Theta^{\mu\nu}$, we are now in a position to evaluate the equal-time commutators of its components. The method is a straightforward application of canonical manipulations, and is very tedious. We quote only the results. We obtain first (always $x_0 = y_0$; $r, s, \ldots$ denote space indices)

$$\mathrm{i}[\Theta^{00}(x), \Theta^{00}(y)] = (\Theta^{0r}(x) + \Theta^{0r}(y))\partial_r^x \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}), \tag{4.1}$$

where there are no more singular terms on the right-hand side; but none are needed. For the second commutator, the result is

$$\begin{aligned}\mathrm{i}[\Theta^{00}(x), \Theta^{0r}(y)]&\\ &= (\Theta^{rs}(x) - \eta^{rs}\Theta^{00}(y))\partial_s^x \delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\\ &\quad + \tfrac{1}{6}\partial_y^r\{[\Box^2\sigma - \partial_\mu\phi^i(2g_{ij} + \phi^l g_{ij,l})\partial^\mu\phi^j]\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})\}\\ &\quad + \tfrac{1}{18}\partial_s^x\partial_t^x\partial_u^y\{[\eta^{st}\eta^{ru}(3\sigma - \phi^i\sigma_{,i}) - 3\eta^{rs}\eta^{tu}\sigma]\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}).\end{aligned} \tag{4.2}$$

The second term on the right-hand side† vanishes on application of the equations of motion. The triple derivative term simplifies when normal coordinates are used, and becomes

$$\tfrac{1}{18}\partial_s^x\partial_t^x\partial_u^y[(\eta^{st}\eta^{ru} - 3\eta^{rs}\eta^{tu})\phi^2\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})], \tag{4.3}$$

which is exactly what was obtained in $\lambda\phi^4$ theory by Callan *et al.* This term is absent in the corresponding expression obtained using $T^{\mu\nu}$, and is needed to satisfy the positivity requirements discussed by Boulware and Deser.

We may again show, even for chiral theory, that (1.10) holds; actually the triple derivative terms in the equal-time commutators give no contribution.

If the equations of motion are used, we have

$$\Box^2\sigma = \partial_\mu\phi^i(2g_{ij} + \phi^l g_{ij,l})\partial^\mu\phi^j, \tag{4.4}$$

and with this we obtain for the trace of $\Theta^{\mu\nu}$

$$\Theta = \tfrac{1}{2}\partial_\mu\phi^i\phi^l g_{ij,l}\partial^\mu\phi^j. \tag{4.5}$$

Now the fields ϕ^l enter into g_{ij} always in the form (ϕ^l/f_π); indeed g_{ij} is an analytic function of this ratio, with coefficients which are pure numbers independent of f_π. This means that we also have

$$\Theta = -\tfrac{1}{2}\partial_\mu\phi^i\left(f_\pi\frac{\partial}{\partial f_\pi}g_{ij}\right)\partial^\mu\phi^j. \tag{4.6}$$

Starting from

$$\Theta^{00} = \tfrac{1}{2}(\pi_i g^{ij}\pi_j + \nabla\phi^i g_{ij}\nabla\phi^j) - \tfrac{1}{6}\nabla^2\sigma \tag{4.7}$$

it is easy then to show

$$f_\pi\frac{\partial}{\partial f_\pi}\Theta^{00} = \Theta. \tag{4.8}$$

Taken in conjunction with (1.10) this again serves as a consistency check because all that (1.10) now means is that Θ^{00} has scale dimension 4.

We note in addition the equal-time commutators for the charge densities

$$\mathrm{i}[\Theta^{00}(x), j_V^{i0}(y)] = j_V^{ir}(x)\partial_r^x\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) \tag{4.9}$$

† We remind the reader that we have *not* paid any attention to the problem of ordering of the noncommuting factors in this expression, or in equation (4.4) below, etc.

for the vector quantities, and (in normal coordinates)

$$\mathrm{i}[\Theta^{00}(x), j_{\mathrm{A}}^{i0}(y)] = j_{\mathrm{A}}^{ir}(x)\partial_r^x\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y}) - \tfrac{1}{3}\nabla_x^2[\phi^i\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})] \tag{4.10}$$

for the axial quantities. The currents are still locally conserved

$$\partial_\mu j^{a\mu} = 0, \tag{4.11}$$

but the energy density Θ^{00} is no longer a *local* chiral invariant. If F^{5i} is a chiral generator, on integrating (4.10) over $\boldsymbol{y}$ we obtain

$$\mathrm{i}[F^{5i}, \Theta^{00}] = \tfrac{1}{3}\nabla^2\phi^i. \tag{4.12}$$

Nonetheless the chiral generators are constants of the motion; equivalently the *total* energy is chiral invariant.

References

Belinfante F J 1940 *Physica* **7** 449–74
Boulware D G and Deser S 1967 *J. math. Phys.* **8** 1468–77
Callan C G Jr, Coleman S and Jackiw R 1970 *Ann. Phys., NY* **59** 42–73
Callan C G Jr, Coleman S, Wess J and Zumino B 1969 *Phys. Rev.* **177** 2247–50
Charap J M 1970 *Can. J. Phys.* **48** 2272–82
Coleman S and Jackiw R 1971 *Ann. Phys., NY* **67** 552–98
Parish C 1973 *J. Phys.* A: *Math., Nucl. Gen.* **6** 1217–23
Weinberg S 1968 *Phys. Rev.* **166** 1568–77